SLIM EXCEPTIONAL SETS AND THE ASYMPTOTIC FORMULA IN WARING'S PROBLEM

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1. Introduction. By avoiding a conventional application of Bessel's inequality in favour of explicitly controlling an exponential sum over the exceptional set, in our previous work [9, 10], we have exploited additional variables so as to enhance exceptional set estimates in various additive problems of Waring type. In this memoir we turn to the problem of establishing the expected asymptotic formula in Waring's problem. The methods introduced herein, although discussed in the context of the asymptotic formula, should nonetheless provide a useful model for future excursions involving exceptional sets in additive problems.

Denote by $R_{s,k}(n)$ the number of representations of a positive integer n as the sum of s kth powers of positive integers. A heuristic application of the circle method suggests that for $k \geqslant 3$ and $s \geqslant k+1$, one should have the asymptotic relation

$$R_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} + o(n^{s/k-1}), \tag{1.1}$$

where

$$\mathfrak{S}_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(q^{-1} \sum_{r=1}^{q} e(ar^k/q) \Big)^s e(-an/q), \tag{1.2}$$

and $e(z)$ denotes $\exp(2\pi i z)$. With the objective of determining how frequently the formula (1.1) might fail, we define an associated exceptional set estimate as follows. When $\psi(t)$ is a function of a positive variable t, denote by $\widetilde{E}_{s,k}(N;\psi)$ the number of integers n with $1 \leqslant n \leqslant N$ for which

$$\Big| R_{s,k}(n) - \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} \Big| > n^{s/k-1} \psi(n)^{-1}.$$

When $\psi(t)$ grows no faster than a suitable power of $\log t$, it follows from work of Vaughan [5, 6] that whenever $s \geqslant 2^k$, one has $\widetilde{E}_{s,k}(N;\psi) \ll 1$ (that is, the expected

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asymptotic formula holds with 2^k or more variables). Incorporating such work into familiar classical methods, it follows that whenever $s \geqslant 2^{k-1}$ and δ is a sufficiently small positive number, one has

$$\widetilde{E}_{s,k}(N;\psi) \ll N^{1-(s2^{2-k}-2)/k}(\log N)^{-\delta}\psi(N)^2. \tag{1.3}$$

Subsequent work of Heath-Brown [4] and Boklan [1] permits the refinement of such estimates for $k \geqslant 6$, and thus one finds that for $s \geqslant 7\cdot 2^{k-3}$ one has $\widetilde{E}_{s,k}(N;\psi) \ll 1$, and indeed one may establish the estimate $\widetilde{E}_{s,k}(N;\psi) = o(N)$ for $s \geqslant 7\cdot 2^{k-4}$. Finally, we comment that methods of Vinogradov in their most modern incarnations (see Wooley [8] and Ford [3]) yield estimates superior to those recorded in (1.3) for $k \geqslant 9$. Indeed, one has $\widetilde{E}_{s,k}(N;\psi) \ll 1$ for $s \geqslant k^2(\log k + \log\log k + O(1))$ and $\widetilde{E}_{s,k}(N;\psi) = o(N)$ for $s \geqslant \frac{1}{2}k^2(\log k + \log\log k + O(1))$.

In previous work devoted to sums of cubes (see Wooley [10]), we established that whenever $\psi(t) = O((\log t)^{1-\delta})$ for some positive number δ, then $\widetilde{E}_{7,3}(N;\psi) \ll_\varepsilon N^{4/9+\varepsilon}$, thereby improving on the classical estimate $\widetilde{E}_{7,3}(N;\psi) \ll N^{1/2}$ available via (1.3). We now consider estimates for $\widetilde{E}_{s,k}(N;\psi)$ for the remaining values of k of interest, beginning in §2 with a discussion of sums of biquadrates. It is convenient here, and elsewhere, to refer to a function $\psi(t)$ as being a *function of uniform growth with exponent* δ, when $\psi(t)$ is a function of a positive variable t, increasing monotonically to infinity, and satisfying the condition that when t is large, one has $\psi(t) = O(t^\delta)$.

Theorem 1.1. *Suppose that $\psi_4(t)$ is a function of uniform growth with exponent δ, for some sufficiently small positive number δ. Then for each positive number ε, one has*

$$\widetilde{E}_{15,4}(N;\psi_4) \ll_\delta N^{7/16+\varepsilon}\psi_4(N)^2.$$

For comparison, the classical bound available via (1.3) yields an estimate marginally sharper than $\widetilde{E}_{15,4}(N;\psi_4) \ll N^{9/16}\psi_4(N)^2$.

In §3 we turn our attention to the asymptotic formula in Waring's problem for fifth powers, and establish the estimates recorded in the following theorem.

Theorem 1.2. *Suppose that $\psi_5(t)$ is a function of uniform growth with exponent δ, for some sufficiently small positive number δ. Then for each positive number ε, one has*

$$\widetilde{E}_{s,5}(N;\psi_5) \ll_\delta N^{\alpha_s+\varepsilon}\psi_5(N)^2 \quad (29 \leqslant s \leqslant 31),$$

where

$$\alpha_{29} = 23/40, \quad \alpha_{30} = 11/20, \quad \alpha_{31} = 3/8.$$

The classical approach leading to (1.3) yields conclusions similar to those recorded in Theorem 1.2, save with $\alpha_{29} = 27/40$, $\alpha_{30} = 13/20$ and $\alpha_{31} = 5/8$.

Moving next, in §4, to sums of sixth powers, we are able to wield the aforementioned work of Heath-Brown and Boklan to good account.

Theorem 1.3. *Suppose that $\psi_6(t)$ is a function of uniform growth with exponent δ, for some sufficiently small positive number δ. Then for each positive number ε, one has*

$$\widetilde{E}_{s,6}(N;\psi_6) \ll_\delta N^{\beta_s+\varepsilon}\psi_6(N)^2 \quad (52 \leqslant s \leqslant 55),$$

where

$$\beta_s = 2/3 - (s-51)/96 \quad (52 \leqslant s \leqslant 55).$$

In §5 we are able to incorporate additional savings for $k = 7$, at least when s is close enough to 112.

Theorem 1.4. *Suppose that $\psi_7(t)$ is a function of uniform growth with exponent δ, for some sufficiently small positive number δ. Then for each positive number ε, one has*

$$\widetilde{E}_{s,7}(N;\psi_7) \ll_\delta N^{\gamma_s+\varepsilon}\psi_7(N)^2 \quad (101 \leqslant s \leqslant 111),$$

where

$$\gamma_s = \begin{cases} 5/7 - (s-100)/224, & \text{when } 101 \leqslant s \leqslant 108, \\ 4/7 - (s-108)/224, & \text{when } 109 \leqslant s \leqslant 111. \end{cases}$$

Finally, in §6 we have at our disposal further resources with which to improve our estimates for exceptional sets associated with eighth powers, especially when s is close to 224.

Theorem 1.5. *Suppose that $\psi_8(t)$ is a function of uniform growth with exponent δ, for some sufficiently small positive number δ. Then for each positive number ε, one has*

$$\widetilde{E}_{s,8}(N;\psi_8) \ll_\delta N^{\delta_s+\varepsilon}\psi_8(N)^2 \quad (197 \leqslant s \leqslant 223),$$

where

$$\delta_s = \begin{cases} 3/4 - (s-196)/512, & \text{when } 197 \leqslant s \leqslant 212, \\ 5/8 - (s-212)/512, & \text{when } 213 \leqslant s \leqslant 220, \\ 1/2 - (s-220)/512, & \text{when } 221 \leqslant s \leqslant 223. \end{cases}$$

Throughout, the letter ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε, unless otherwise indicated. In an effort to simplify our analysis, we adopt the convention that whenever ε appears in a statement, then we are implicitly asserting that for each $\varepsilon > 0$ the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. Finally, we write $[z]$ to denote the largest integer not exceeding z.

2. Sums of biquadrates. We begin with an account of the proof of Theorem 1.1, this permitting the introduction of notation and auxiliary estimates of utility both here and hereafter. We model the initial stages of our argument on the framework introduced by Brüdern, Kawada and Wooley [2] in their work on exceptional sets in thin sequences. For the moment, we consider arbitrary integers k and s with $4 \leqslant k \leqslant 8$ and $s \geqslant 3 \cdot 2^{k-2}$, though later in this section we specialise to the case $k = 4$ and $s = 15$. Let N be a large positive number, and let $\psi = \psi_k(t)$ be a function of the type described in the statements of Theorems 1.1–1.5. We denote by $\mathcal{Z}_{s,k}(N)$ the set of integers n with $N/2 < n \leqslant N$ for which the inequality

$$\left| R_{s,k}(n) - \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} \right| > n^{s/k-1} \psi_k(n)^{-1} \tag{2.1}$$

holds, and we abbreviate $\mathrm{card}(\mathcal{Z}_{s,k}(N))$ to $Z_{s,k}$.

Write $P_k = [N^{1/k}]$ and define

$$f_k(\alpha) = \sum_{1 \leqslant x \leqslant P_k} e(\alpha x^k).$$

Then by orthogonality, for each integer n with $N/2 < n \leqslant N$ one has

$$R_{s,k}(n) = \int_0^1 f_k(\alpha)^s e(-n\alpha)\, d\alpha. \tag{2.2}$$

Let $\mathfrak{M} = \mathfrak{M}_k$ denote the union of the intervals

$$\mathfrak{M}_k(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant (2k)^{-1} P_k N^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant (2k)^{-1} P_k$ and $(a,q) = 1$. Then it follows from Theorem 4.4 of Vaughan [7] that whenever $N/2 < n \leqslant N$, one has

$$\int_{\mathfrak{M}} f_k(\alpha)^s e(-n\alpha)\, d\alpha = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} + O(n^{s/k-1-2\delta}), \tag{2.3}$$

where $\mathfrak{S}_{s,k}(n)$ denotes the singular series defined in (1.2). Note here our use of the implicit assumption that δ is a sufficiently small positive number. Now define $\mathfrak{m} = \mathfrak{m}_k$ by writing $\mathfrak{m}_k = [0,1) \setminus \mathfrak{M}_k$. Then for $n \in \mathcal{Z}_{s,k}(N)$, it follows from (2.1), (2.2), (2.3) and our assumed upper bound $\psi_k(t) = O(t^\delta)$, that

$$\left| \int_{\mathfrak{m}} f_k(\alpha)^s e(-n\alpha)\, d\alpha \right| > \tfrac{1}{2} n^{s/k-1} \psi_k(n)^{-1}. \tag{2.4}$$

Define the complex number $\eta_n = \eta_n(s,k)$ by taking $\eta_n = 0$ for $n \notin \mathcal{Z}_{s,k}(N)$, and when $n \in \mathcal{Z}_{s,k}(N)$ by means of the equation

$$\left| \int_{\mathfrak{m}} f_k(\alpha)^s e(-n\alpha)\, d\alpha \right| = \eta_n(s,k) \int_{\mathfrak{m}} f_k(\alpha)^s e(-n\alpha)\, d\alpha.$$

Plainly, one has $|\eta_n| = 1$ whenever η_n is non-zero. Thus, it follows from (2.4) that

$$\begin{aligned} N^{s/k-1} \psi_k(N)^{-1} \mathrm{card}(\mathcal{Z}_{s,k}(N)) &\ll \sum_{N/2 < n \leqslant N} \eta_n \int_{\mathfrak{m}} f_k(\alpha)^s e(-n\alpha)\, d\alpha \\ &= \int_{\mathfrak{m}} f_k(\alpha)^s K_{s,k}(-\alpha)\, d\alpha, \end{aligned} \tag{2.5}$$

where the exponential sum $K_{s,k}(\alpha)$ is defined by

$$K_{s,k}(\alpha)=\sum_{N/2<n\leqslant N}\eta_n(s,k)e(n\alpha). \tag{2.6}$$

Our strategy is to estimate the integral on the right hand side of (2.5), and thereby obtain an upper bound for $Z_{s,k}$. Before advancing towards this goal in the particular case in which $k=4$ and $s=15$, we introduce an auxiliary mean value estimate of use also in subsequent sections.

Lemma 2.1. *Suppose that $k\geqslant 2$, that $1\leqslant j\leqslant k-1$, and that $\varepsilon>0$. Then one has*

$$\int_0^1|f_k(\alpha)^{2^j}K_{s,k}(\alpha)^2|d\alpha\ll P_k^{2^j+\varepsilon}(P_k^{-j-1}Z_{s,k}^2+P_k^{-1}Z_{s,k}). \tag{2.7}$$

Proof. Let Δ_j denote the jth iterate of the forward differencing operator, so that whenever ϕ is a function of a real variable z, one has

$$\Delta_1(\phi(z);h)=\phi(z+h)-\phi(z),$$

and when $J\geqslant 1$,

$$\Delta_{J+1}(\phi(z);h_1,\dots,h_{J+1})=\Delta_1(\Delta_J(\phi(z);h_1,\dots,h_J);h_{J+1}).$$

It follows via a modest computation that

$$\Delta_J(z^k;\mathbf{h})=h_1\dots h_Jp_J(z;\mathbf{h}),$$

where p_J is a homogeneous polynomial in z and $\mathbf{h}$ of total degree $k-J$, in which the coefficient of z^{k-J} is $k!/(k-J)!$. By the Weyl differencing lemma (see, for example, Lemma 2.3 of Vaughan [7]), one has

$$|f_k(\alpha)|^{2^j}\leqslant(2P_k)^{2^j-j-1}\sum_{|h_1|<P_k}\cdots\sum_{|h_j|<P_k}T_j,$$

where

$$T_j=\sum_{x\in I_j}e(\alpha h_1\dots h_jp_j(x;\mathbf{h})),$$

and $I_j=I_j(\mathbf{h})$ denotes an interval of integers, possibly empty, contained in $[1,P_k]$. On recalling (2.6), therefore, it follows from orthogonality that the integral on the left hand side of (2.7) is bounded above by the number of integral solutions of the equation

$$h_1\dots h_jp_j(z;\mathbf{h})=n_1-n_2, \tag{2.8}$$

with $|h_i|<P_k$ $(1\leqslant i\leqslant j)$, $1\leqslant z\leqslant P_k$ and $n_l\in\mathcal{Z}_{s,k}(N)$ $(l=1,2)$, and with each solution being counted with weight

$$(2P_k)^{2^j-j-1}.$$

Consider a solution z, $\mathbf{h}$, $\mathbf{n}$ of the equation (2.8) satisfying the associated conditions. There are plainly $O(P_k^{j-1})$ choices of $\mathbf{h}$ in which one at least of the h_i is zero, and in such circumstances one necessarily has $n_1 = n_2$. Given any one of the $O(Z_{s,k}^2)$ possible choices for n_1 and n_2 with $n_1 \neq n_2$, meanwhile, an elementary divisor function estimate shows that there are $O(P_k^\varepsilon)$ permissible choices of z and $\mathbf{h}$ satisfying (2.8). We thus deduce that the total number, $\mathcal{T}_0$, of solutions of (2.8) satisfies

$$\mathcal{T}_0 \ll P_k^j Z_{s,k} + P_k^\varepsilon Z_{s,k}^2.$$

Consequently, on recalling the weights associated with our upper bound for the integral on the left hand side of (2.7), we conclude that

$$\int_0^1 |f_k(\alpha)^{2^j} K_{s,k}(\alpha)^2| d\alpha \ll P_k^{2^j-j-1}(P_k^j Z_{s,k} + P_k^\varepsilon Z_{s,k}^2).$$

The conclusion of the lemma follows immediately.

We now restrict attention to the situation in which $k = 4$ and $s = 15$, and establish the estimate for $\widetilde{E}_{15,4}(N;\psi_4)$ claimed in the statement of Theorem 1.1. Observe first that by applying Schwarz's inequality on the right hand side of (2.5), we obtain the inequality

$$N^{11/4}\psi_4(N)^{-1} Z_{15,4} \ll I_1^{1/2} I_2^{1/2}, \tag{2.9}$$

where

$$I_1 = \int_{\mathfrak{m}} |f_4(\alpha)|^{26} d\alpha$$

and

$$I_2 = \int_0^1 |f_4(\alpha)^4 K_{15,4}(\alpha)^2| d\alpha.$$

But Weyl's inequality (see, for example, Lemma 2.4 of Vaughan [7]) yields the upper bound

$$\sup_{\alpha\in\mathfrak{m}} |f_4(\alpha)| \ll P_4^{7/8+\varepsilon},$$

and Hua's lemma (see Lemma 2.5 of Vaughan [7]) establishes that

$$\int_0^1 |f_4(\alpha)|^{16} d\alpha \ll P_4^{12+\varepsilon}.$$

Thus we deduce that

$$\begin{aligned} I_1 &\ll \Big(\sup_{\alpha\in\mathfrak{m}} |f_4(\alpha)|\Big)^{10} \int_0^1 |f_4(\alpha)|^{16} d\alpha \\ &\ll (P_4^{7/8+\varepsilon})^{10} P_4^{12+\varepsilon} \ll P_4^{83/4+\varepsilon}. \end{aligned}$$

On substituting the latter estimate into (2.9), and applying Lemma 2.1 with $j=2$, we find that

$$N^{11/4}\psi_4(N)^{-1}Z_{15,4} \ll N^{83/32+\varepsilon}(N^{1/4}Z_{15,4}^2 + N^{3/4}Z_{15,4})^{1/2},$$

whence

$$Z_{15,4} \ll Z_{15,4}\psi_4(N)N^{\varepsilon-1/32} + Z_{15,4}^{1/2}\psi_4(N)N^{7/32+\varepsilon}.$$

Thus, on recalling that for some sufficiently small positive number δ one has $\psi_4(t) = O(t^{\delta})$, it follows that

$$Z_{15,4} \ll N^{7/16+\varepsilon}\psi_4(N)^2,$$

and the conclusion of Theorem 1.1 follows by summing over dyadic intervals.

3. Sums of fifth powers. The proof of Theorem 1.2 may be completed in two of the three cases by adjusting the argument applied in the previous section. The third case considered in Theorem 1.2, however, requires a new mean value estimate similar to that provided by Lemma 2.1.

Lemma 3.1. *Suppose that $k \geqslant 3$, that $2 \leqslant j \leqslant k-1$, and that $\varepsilon > 0$. Then one has*

$$\int_0^1 |f_k(\alpha)^{3\cdot 2^{j-1}} K_{s,k}(\alpha)^2|d\alpha \ll P_k^{3\cdot 2^{j-1}+\varepsilon}(P_k^{-j-1}Z_{s,k}^2 + P_k^{-2}Z_{s,k}). \tag{3.1}$$

Proof. Applying Weyl differencing as in the proof of Lemma 2.1, we find by orthogonality that in this instance, the integral on the left hand side of (3.1) is bounded above by the number of integral solutions of the equation

$$h_1 \dots h_j p_j(z;\mathbf{h}) = n_1 - n_2 + \sum_{i=1}^{2^{j-2}} (x_i^k - y_i^k), \tag{3.2}$$

with $|h_l| < P_k$ $(1 \leqslant l \leqslant j)$, $1 \leqslant z \leqslant P_k$, $1 \leqslant x_i, y_i \leqslant P_k$ $(1 \leqslant i \leqslant 2^{j-2})$ and $n_m \in \mathcal{Z}_{s,k}(N)$ $(m=1,2)$, and with each solution being counted with weight

$$(2P_k)^{2^j-j-1}.$$

Consider a solution $z, \mathbf{h}, \mathbf{x}, \mathbf{y}, \mathbf{n}$ of the equation (3.2) satisfying the associated conditions. There are plainly $O(P_k^{j-1})$ choices of $\mathbf{h}$ in which one at least of the h_l is zero, and in such circumstances one has

$$\sum_{i=1}^{2^{j-2}} (x_i^k - y_i^k) = n_2 - n_1. \tag{3.3}$$

By orthogonality, it follows that the number, $\mathcal{T}_1$, of such solutions is at most

$$P_k^j \int_0^1 |f_k(\alpha)^{2^{j-1}} K_{s,k}(\alpha)^2|d\alpha.$$

Consequently, we find from Lemma 2.1 that

$$\mathcal{T}_1 \ll P_k^{2^{j-1}+j+\varepsilon}(P_k^{-j}Z_{s,k}^2 + P_k^{-1}Z_{s,k}). \tag{3.4}$$

Given any one of the

$$O(Z_{s,k}^2 P_k^{2^{j-1}})$$

possible choices of $\mathbf{n}, \mathbf{x}, \mathbf{y}$ for which the equation (3.3) does not hold, meanwhile, an elementary divisor function estimate shows that there are $O(P_k^\varepsilon)$ permissible choices of z and $\mathbf{h}$ satisfying (3.2). We thus deduce that the number, $\mathcal{T}_2$, of solutions of this second type satisfies

$$\mathcal{T}_2 \ll P_k^{2^{j-1}+\varepsilon} Z_{s,k}^2. \tag{3.5}$$

On recalling the weights associated with our upper bound for the integral on the left hand side of (3.1), we conclude from (3.4) and (3.5) that

$$\begin{aligned}\int_0^1 |f_k(\alpha)^{3\cdot 2^{j-1}} K_{s,k}(\alpha)^2| d\alpha &\ll P_k^{2^j-j-1}(\mathcal{T}_1 + \mathcal{T}_2)\\ &\ll P_k^{3\cdot 2^{j-1}-1+\varepsilon}(P_k^{-j}Z_{s,k}^2 + P_k^{-1}Z_{s,k}).\end{aligned}$$

The conclusion of the lemma is now immediate.

We initiate our proof of Theorem 1.2 by recalling the inequality (2.5). Let s be an integer with $s \geqslant 29$, and apply Schwarz's inequality to (2.5) to obtain the upper bound

$$N^{(s-5)/5}\psi_5(N)^{-1} Z_{s,5} \ll I_3^{1/2} I_4^{1/2}, \tag{3.6}$$

where

$$I_3 = \int_{\mathfrak{m}} |f_5(\alpha)|^{2s-8} d\alpha$$

and

$$I_4 = \int_0^1 |f_5(\alpha)^8 K_{s,5}(\alpha)^2| d\alpha.$$

But Weyl's inequality yields the upper bound

$$\sup_{\alpha \in \mathfrak{m}} |f_5(\alpha)| \ll P_5^{15/16+\varepsilon},$$

and Hua's lemma reveals that

$$\int_0^1 |f_5(\alpha)|^{32} d\alpha \ll P_5^{27+\varepsilon},$$

and thus we obtain

$$\begin{aligned}I_3 &\ll \Big(\sup_{\alpha\in\mathfrak{m}} |f_5(\alpha)|\Big)^{2s-40} \int_0^1 |f_5(\alpha)|^{32} d\alpha\\ &\ll P_5^{2s-13-(s-20)/8+\varepsilon}.\end{aligned} \tag{3.7}$$

On substituting the latter estimate into (3.6), and applying Lemma 2.1 with $j = 3$, we arrive at the upper bound

$$N^{(s-5)/5}\psi_5(N)^{-1}Z_{s,5} \ll N^{(s-5/2)/5-(s-20)/80+\varepsilon}(N^{-4/5}Z_{s,5}^2 + N^{-1/5}Z_{s,5})^{1/2},$$

whence

$$Z_{s,5} \ll Z_{s,5}\psi_5(N)N^{\varepsilon-(s-28)/80} + Z_{s,5}^{1/2}\psi_5(N)N^{\varepsilon-(s-52)/80}. \tag{3.8}$$

On recalling that for some sufficiently small positive number δ one has $\psi_5(t) = O(t^\delta)$, it follows from (3.8) with $s = 29$ that

$$Z_{29,5} \ll N^{23/40+\varepsilon}\psi_5(N)^2, \tag{3.9}$$

and similarly with $s = 30$, we find that

$$Z_{30,5} \ll N^{11/20+\varepsilon}\psi_5(N)^2. \tag{3.10}$$

When $s = 31$ we proceed along a slightly different path, now applying Schwarz's inequality to (2.5) in the shape

$$N^{26/5}\psi_5(N)^{-1}Z_{31,5} \ll I_5^{1/2}I_6^{1/2}, \tag{3.11}$$

where

$$I_5 = \int_{\mathfrak{m}} |f_5(\alpha)|^{50} d\alpha$$

and

$$I_6 = \int_0^1 |f_5(\alpha)^{12}K_{31,5}(\alpha)^2| d\alpha.$$

Here the estimate

$$I_5 \ll P_5^{351/8+\varepsilon}$$

follows via the argument leading to (3.7), and hence on substituting this estimate into (3.11), and applying Lemma 3.1 with $j = 3$, we deduce that

$$N^{26/5}\psi_5(N)^{-1}Z_{31,5} \ll N^{447/80+\varepsilon}(N^{-4/5}Z_{31,5}^2 + N^{-2/5}Z_{31,5})^{1/2}.$$

Consequently, one has

$$Z_{31,5} \ll Z_{31,5}\psi_5(N)N^{\varepsilon-1/80} + Z_{31,5}^{1/2}\psi_5(N)N^{3/16+\varepsilon}.$$

Then on recalling that for some sufficiently small positive number δ one has $\psi_5(t) = O(t^\delta)$, one may conclude that

$$Z_{31,5} \ll N^{3/8+\varepsilon}\psi_5(N)^2. \tag{3.12}$$

The proof of Theorem 1.2 is completed by collecting together (3.9), (3.10) and (3.12), and summing over dyadic intervals.

4. Sums of sixth powers. We turn our attention next to the proof of Theorem 1.3, and this entails the use of technology introduced by Heath-Brown [4] for the investigation of Waring's problem for kth powers, with $k \geqslant 6$. Since these new estimates will be crucial also in our analysis relevant to seventh and eighth powers, we take the liberty of providing a somewhat general analysis. An account of such a treatment has the potential to encompass much space, and here we economise by referring closely to an account due to Boklan [1] devoted to a slightly more precise analysis, the relevant details being more easily extracted from this account.

Our starting point in this discussion is again the upper bound (2.5), but instead of handling the minor arcs $\mathfrak{m}$ directly, we employ a somewhat more sophisticated analysis based on the argument of Heath-Brown [4] as sharpened by Boklan [1]. In this context, when r is a non-negative integer, we write

$$\Omega_{r,k} = \int_0^1 |f_k(\alpha)^{2r} K_{s,k}(\alpha)^2| d\alpha.$$

Lemma 4.1. *Suppose that $k \geqslant 6$, and that s, t, u, v, w are non-negative integers with*

$$s = 7 \cdot 2^{k-4} + t + u \quad \textit{and} \quad s = 3 \cdot 2^{k-3} + [(k+1)/2] + v + w.$$

Then for each $\varepsilon > 0$ one has

$$\int_{\mathfrak{m}} |f_k(\alpha)^s K_{s,k}(\alpha)| d\alpha \ll P_k^{7\cdot 2^{k-4} - k/2 + \varepsilon} \left(P_k^{1-(8/3)2^{-k}} \right)^t \Omega_{u,k}^{1/2} + P_k^{3\cdot 2^{k-3} + [(k+1)/2] - k/2 + \varepsilon} (P_k^{1-2^{1-k}})^v \Omega_{w,k}^{1/2}.$$

Proof. Define the narrow set of minor arcs $\mathfrak{n}$ to be the set of numbers $\alpha \in [0,1)$ with the property that whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$ and $|q\alpha - a| \leqslant P_k^{3-k}$, then one has $q > \frac{1}{4} P_k^3$. The work of Heath-Brown [4] (see Theorem 1 of [4], or equation (6.4) of Boklan [1]) shows that

$$\sup_{\alpha \in \mathfrak{n}} |f_k(\alpha)| \ll P_k^{1-(8/3)2^{-k}+\varepsilon}. \tag{4.1}$$

Moreover, Theorem 2 of Heath-Brown [4] shows that

$$\int_0^1 |f_k(\alpha)|^{7\cdot 2^{k-3}} d\alpha \ll P_k^{7\cdot 2^{k-3} - k + \varepsilon}. \tag{4.2}$$

An application of Schwarz's inequality in combination with (4.1) and (4.2) therefore reveals that

$$\begin{aligned} \int_{\mathfrak{n}} |f_k(\alpha)^s K_{s,k}(\alpha)| d\alpha &\leqslant \Big(\sup_{\alpha \in \mathfrak{n}} |f_k(\alpha)| \Big)^t \Big(\int_0^1 |f_k(\alpha)|^{7\cdot 2^{k-3}} d\alpha \Big)^{1/2} \Omega_{u,k}^{1/2} \\ &\ll (P_k^{1-(8/3)2^{-k}+\varepsilon})^t (P_k^{7\cdot 2^{k-3}-k+\varepsilon})^{1/2} \Omega_{u,k}^{1/2}. \end{aligned} \tag{4.3}$$

The analysis of the set $\mathfrak{N} = \mathfrak{m} \setminus \mathfrak{n}$ complementary to $\mathfrak{n}$ involves a division into sets $I(q, Z_v)$, with a classification of available numbers q according to membership in a set Υ_v, the precise definitions of which need not concern us here. For the application at hand we may be expedient and extract only the information relevant to our needs, ignoring superfluous details. Thus one finds that when α belongs to the subset $\mathfrak{N}_1$ of $\mathfrak{N}$ corresponding to the sets $I(q, Z_v)$ with $q \notin \Upsilon_v$, one has

$$|f_k(\alpha)| \ll P_k^{1-(8/3)2^{-k}+\varepsilon}$$

(see equation (8.3) of Boklan [1]). A comparison with (4.1) leads from here to the estimate

$$\int_{\mathfrak{N}_1} |f_k(\alpha)^s K_{s,k}(\alpha)| d\alpha \ll (P_k^{1-(8/3)2^{-k}+\varepsilon})^t (P_k^{7\cdot 2^{k-3}-k+\varepsilon})^{1/2} \Omega_{u,k}^{1/2}, \tag{4.4}$$

via an argument parallel to the deduction of (4.3) from (4.1). In order to estimate the contribution arising from the complementary set $\mathfrak{N}_2 = \mathfrak{N} \setminus \mathfrak{N}_1$, meanwhile, one may make use of the argument of Boklan [1] leading to equation (10.2), and ultimately (10.3), of that paper. Thus one deduces that

$$\int_{\mathfrak{N}_2} |f_k(\alpha)|^{6\cdot 2^{k-3}+2[(k+1)/2]+2v} d\alpha \ll (P_k^{1-2^{1-k}+\varepsilon})^{2v} P_k^{6\cdot 2^{k-3}+2[(k+1)/2]-k+\varepsilon}.$$

An application of Schwarz's inequality consequently yields the upper bound

$$\begin{aligned}\int_{\mathfrak{N}_2} |f_k(\alpha)^s K_{s,k}(\alpha)| d\alpha &\leqslant \Big(\int_{\mathfrak{N}_2} |f_k(\alpha)|^{6\cdot 2^{k-3}+2[(k+1)/2]+2v} d\alpha \Big)^{1/2} \Omega_{w,k}^{1/2} \\ &\ll (P_k^{1-2^{1-k}+\varepsilon})^v P_k^{3\cdot 2^{k-3}+[(k+1)/2]-k/2} \Omega_{w,k}^{1/2}. \end{aligned} \tag{4.5}$$

Since, plainly, the set $\mathfrak{m}$ is the union of $\mathfrak{n}$, $\mathfrak{N}_1$ and $\mathfrak{N}_2$, the conclusion of the lemma is immediate from (4.3), (4.4) and (4.5).

We now return to the proof of Theorem 1.3, wherein we suppose that $k = 6$. Observe first that when h is a positive integer and $s = 51+h$, we may apply Lemma 4.1 with $t = 15 + h$, $v = 16 + h$ and $u = w = 8$ in order to obtain the upper bound

$$\int_{\mathfrak{m}} |f_6(\alpha)^s K_{s,6}(\alpha)| d\alpha \ll P_6^{79/2+(31/32)h+\varepsilon} \Omega_{8,6}^{1/2}.$$

Consequently, on making use of Lemma 2.1 with $j = 4$ in order to estimate $\Omega_{8,6}$, we find from (2.5) that

$$N^{(45+h)/6} \psi_6(N)^{-1} Z_{s,6} \ll N^{(95+2h)/12-h/192+\varepsilon} (N^{-5/6} Z_{s,6}^2 + N^{-1/6} Z_{s,6})^{1/2}.$$

We therefore deduce that

$$Z_{s,6} \ll Z_{s,6} \psi_6(N) N^{\varepsilon-h/192} + Z_{s,6}^{1/2} \psi_6(N) N^{1/3-h/192+\varepsilon}.$$

On recalling that for some sufficiently small positive number δ one has $\psi_6(t) = O(t^\delta)$, it follows that

$$Z_{51+h,6} \ll N^{2/3-h/96+\varepsilon} \psi_6(N)^2,$$

and the proof of Theorem 1.3 follows on summing over dyadic intervals.

5. Sums of seventh powers. The argument required to dispose of the proof of Theorem 1.4 requires no tools beyond those developed already in §§2–4, and so we launch our proof immediately. Suppose first that h is a positive integer and $s = 100 + h$. On taking $t = 28 + h$, $v = 32 + h$ and $u = w = 16$, we deduce from Lemma 4.1 that

$$\int_{\mathfrak{m}} |f_7(\alpha)^s K_{s,7}(\alpha)|\, d\alpha \ll P_7^{80+(63/64)h+\varepsilon}\Omega_{16,7}^{1/2}.$$

Applying Lemma 2.1 with $j = 5$ to provide an upper bound for $\Omega_{16,7}$, we find from (2.5) that

$$N^{(93+h)/7}\psi_7(N)^{-1}Z_{s,7} \ll N^{(96+h)/7-h/448+\varepsilon}(N^{-6/7}Z_{s,7}^2 + N^{-1/7}Z_{s,7})^{1/2},$$

whence

$$Z_{s,7} \ll Z_{s,7}\psi_7(N)N^{\varepsilon-h/448} + Z_{s,7}^{1/2}\psi_7(N)N^{5/14-h/448+\varepsilon}.$$

On recalling that for some sufficiently small positive number δ one has $\psi_7(t) = O(t^\delta)$, we conclude that

$$Z_{100+h,7} \ll N^{5/7-h/224+\varepsilon}\psi_7(N)^2.$$

The conclusion of Theorem 1.4 is therefore immediate for $101 \leqslant s \leqslant 108$ on summing over dyadic intervals.

Suppose next that h is a positive integer and $s = 108+h$. We now put $t = 28+h$, $v = 32 + h$ and $u = w = 24$, and conclude from Lemma 4.1 that

$$\int_{\mathfrak{m}} |f_7(\alpha)^s K_{s,7}(\alpha)|\, d\alpha \ll P_7^{80+(63/64)h+\varepsilon}\Omega_{24,7}^{1/2}.$$

On this occasion we apply Lemma 3.1 with $j = 5$ to provide an upper bound for $\Omega_{24,7}$, and hence deduce from (2.5) that

$$N^{(101+h)/7}\psi_7(N)^{-1}Z_{s,7} \ll N^{(104+h)/7-h/448+\varepsilon}(N^{-6/7}Z_{s,7}^2 + N^{-2/7}Z_{s,7})^{1/2}.$$

Consequently, one obtains

$$Z_{s,7} \ll Z_{s,7}\psi_7(N)N^{\varepsilon-h/448} + Z_{s,7}^{1/2}\psi_7(N)N^{2/7-h/448+\varepsilon}.$$

On recalling that for some sufficiently small positive number δ one has $\psi_7(t) = O(t^\delta)$, we infer that

$$Z_{108+h,7} \ll N^{4/7-h/224+\varepsilon}\psi_7(N)^2.$$

The conclusion of Theorem 1.4 for $109 \leqslant s \leqslant 111$ now follows on summing over dyadic intervals.

6. Sums of eighth powers. Before embarking on the proof of Theorem 1.5, we complete our arsenal of auxiliary lemmata with one final estimate of flavour similar to Lemmata 2.1 and 3.1.

Lemma 6.1. *Suppose that $k \geqslant 4$, that $3 \leqslant j \leqslant k-1$, and that $\varepsilon > 0$. Then one has*

$$\int_0^1 |f_k(\alpha)^{7\cdot 2^{j-2}} K_{s,k}(\alpha)^2| d\alpha \ll P_k^{7\cdot 2^{j-2}+\varepsilon}(P_k^{-j-1} Z_{s,k}^2 + P_k^{-3} Z_{s,k}). \tag{6.1}$$

Proof. We apply Weyl differencing as in the proof of Lemma 2.1, and conclude from orthogonality that the integral on the left hand side of (6.1) is bounded above by the number of integral solutions of the equation

$$h_1 \dots h_j p_j(z; \mathbf{h}) = n_1 - n_2 + \sum_{i=1}^{3\cdot 2^{j-3}} (x_i^k - y_i^k), \tag{6.2}$$

with $|h_l| < P_k$ $(1 \leqslant l \leqslant j)$, $1 \leqslant z \leqslant P_k$, $1 \leqslant x_i, y_i \leqslant P_k$ $(1 \leqslant i \leqslant 3 \cdot 2^{j-3})$ and $n_m \in \mathcal{Z}_{s,k}(N)$ $(m = 1, 2)$, and with each solution being counted with weight

$$(2P_k)^{2^j - j - 1}.$$

Consider a solution $z, \mathbf{h}, \mathbf{x}, \mathbf{y}, \mathbf{n}$ of the equation (6.2) satisfying the associated conditions. There are $O(P_k^{j-1})$ choices of $\mathbf{h}$ in which one at least of the h_l is zero, and then one has

$$\sum_{i=1}^{3\cdot 2^{j-3}} (x_i^k - y_i^k) = n_2 - n_1. \tag{6.3}$$

By orthogonality, the number, $\mathcal{T}_3$, of such solutions is at most

$$P_k^j \int_0^1 |f_k(\alpha)^{3\cdot 2^{j-2}} K_{s,k}(\alpha)^2| d\alpha.$$

Then Lemma 3.1 yields the upper bound

$$\mathcal{T}_3 \ll P_k^{3\cdot 2^{j-2}+j+\varepsilon}(P_k^{-j} Z_{s,k}^2 + P_k^{-2} Z_{s,k}). \tag{6.4}$$

Given any one of the $O(Z_{s,k}^2 P_k^{3\cdot 2^{j-2}})$ possible choices of $\mathbf{n}, \mathbf{x}, \mathbf{y}$ for which (6.3) does not hold, on the other hand, a divisor function estimate shows that there are $O(P_k^{\varepsilon})$ permissible choices of z and $\mathbf{h}$ satisfying (6.2). Consequently, the number, $\mathcal{T}_4$, of solutions of this type satisfies

$$\mathcal{T}_4 \ll P_k^{3\cdot 2^{j-2}+\varepsilon} Z_{s,k}^2. \tag{6.5}$$

Reintroducing the weights associated with our upper bound for the integral on the left hand side of (6.1), we find from (6.4) and (6.5) that

$$\begin{aligned}\int_0^1 |f_k(\alpha)^{7\cdot 2^{j-2}} K_{s,k}(\alpha)^2|d\alpha &\ll P_k^{2^j-j-1}(\mathcal{T}_3+\mathcal{T}_4)\\ &\ll P_k^{7\cdot 2^{j-2}-1+\varepsilon}(P_k^{-j}Z_{s,k}^2+P_k^{-2}Z_{s,k}).\end{aligned}$$

The conclusion of the lemma now follows immediately.

Our starting point in the proof of Theorem 1.5 is again the inequality (2.5), though of course we may now suppose that $k=8$. We take h to be a positive integer, and put $s=196+h$. Then applying Lemma 4.1 with $t=52+h$, $v=64+h$ and $u=w=32$, we find that

$$\int_{\mathfrak{m}} |f_8(\alpha)^s K_{s,8}(\alpha)|d\alpha \ll P_8^{319/2+(127/128)h+\varepsilon}\Omega_{32,8}^{1/2}.$$

Applying Lemma 2.1 with $j=6$ to provide an upper bound for $\Omega_{32,8}$, we find from (2.5) that

$$N^{(188+h)/8}\psi_8(N)^{-1}Z_{s,8} \ll N^{(383+2h)/16-h/1024+\varepsilon}(N^{-7/8}Z_{s,8}^2+N^{-1/8}Z_{s,8})^{1/2},$$

whence

$$Z_{s,8} \ll Z_{s,8}\psi_8(N)N^{\varepsilon-h/1024}+Z_{s,8}^{1/2}\psi_8(N)N^{3/8-h/1024+\varepsilon}.$$

On recalling that for some sufficiently small positive number δ one has $\psi_8(t)=O(t^\delta)$, we conclude that

$$Z_{196+h,8} \ll N^{3/4-h/512+\varepsilon}\psi_8(N)^2,$$

and the conclusion of Theorem 1.5 follows for $197\leqslant s\leqslant 212$ on summing over dyadic intervals.

Suppose next that h is a positive integer and $s=212+h$. We now put $t=52+h$, $v=64+h$ and $u=w=48$, and discover from Lemma 4.1 that

$$\int_{\mathfrak{m}} |f_8(\alpha)^s K_{s,8}(\alpha)|d\alpha \ll P_8^{319/2+(127/128)h+\varepsilon}\Omega_{48,8}^{1/2}.$$

Applying now Lemma 3.1 with $j=6$ to provide an upper bound for $\Omega_{48,8}$, we conclude from (2.5) that

$$N^{(204+h)/8}\psi_8(N)^{-1}Z_{s,8} \ll N^{(415+2h)/16-h/1024+\varepsilon}(N^{-7/8}Z_{s,8}^2+N^{-1/4}Z_{s,8})^{1/2},$$

and from this we infer that

$$Z_{s,8} \ll Z_{s,8}\psi_8(N)N^{\varepsilon-h/1024}+Z_{s,8}^{1/2}\psi_8(N)N^{5/16-h/1024+\varepsilon}.$$

On recalling again that for some sufficiently small positive number δ one has $\psi_8(t) = O(t^\delta)$, we may again conclude that

$$Z_{212+h,8} \ll N^{5/8-h/512+\varepsilon}\psi_8(N)^2,$$

and this establishes the conclusion of Theorem 1.5 for $213 \leqslant s \leqslant 220$, on summing over dyadic intervals.

Finally, suppose that h is a positive integer and $s = 220 + h$. We now put $t = 52 + h$, $v = 64 + h$ and $u = w = 56$, and discern from Lemma 4.1 that

$$\int_{\mathfrak{m}} |f_8(\alpha)^s K_{s,8}(\alpha)| d\alpha \ll P_8^{319/2+(127/128)h+\varepsilon}\Omega_{56,8}^{1/2}.$$

Wielding Lemma 6.1 with $j = 6$ on this occasion to provide an upper bound for $\Omega_{56,8}$, we deduce from (2.5) that

$$N^{(212+h)/8}\psi_8(N)^{-1}Z_{s,8} \ll N^{(431+2h)/16-h/1024+\varepsilon}(N^{-7/8}Z_{s,8}^2 + N^{-3/8}Z_{s,8})^{1/2},$$

and from this we infer that

$$Z_{s,8} \ll Z_{s,8}\psi_8(N)N^{\varepsilon-h/1024} + Z_{s,8}^{1/2}N^{1/4-h/1024+\varepsilon}.$$

Thus, on recalling that there is a sufficiently small positive number δ for which $\psi_8(t) = O(t^\delta)$, we can conclude that

$$Z_{220+h,8} \ll N^{1/2-h/512+\varepsilon}\psi_8(N)^2.$$

The conclusion of Theorem 1.5 follows for $221 \leqslant s \leqslant 223$ on summing over dyadic intervals, and thus the proof of Theorem 1.5 is at last complete in all details.

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